INTEGRAL CONDITIONS IN THE THEORY OF THE BELTRAMI EQUATIONS

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DEDICATED TO 85 YEARS OF OLLI LEHTO

Abstract

It is shown that many recent and new results on the existence of ACL homeomorphic solutions for the degenerate Beltrami equations with integral constraints follow from our extension of the well–known Lehto existence theorem.

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1 Introduction

The classical case was investigated long ago, see e.g. [Ah], [Bel], [Boj] and [LV]. The existence problem for degenerate Beltrami equations is currently an active area of research. It has been studied extensively and many contributions have been made, see e.g. [AIM], [BGR_1]–[BGR_2], [BJ_1]–[BJ_2], [Ch], [Da], [GMSV_1]–[GMSV_2], [IM_1]–[IM_2], [Kr], [Le], [MM], [MMV], [MRSY], [MS], [Pe], [Tu], [RSY_1]–[RSY_6] and [Ya], see also the survey [SY]. The goal here is to show that our extension of the Lehto existence theorem has as corollaries the main known existence theorems as well as a series of more advanced theorems for the Beltrami equations, see Section 4. The base for these advances is some lemmas on integral conditions in Sections 2 and 3. Then we show in Section 5 that the integral conditions found in Section 4 are not only sufficient but also necessary in the existence theorems for the Beltrami equations with integral restrictions. Finally, the corresponding historic comments and final remarks can be found in Section 6.

Let D be a domain in the complex plane $\mathbb{C}$, i.e., a connected open subset of $\mathbb{C}$, and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. The **Beltrami equation** is

$$f_{\overline{z}} = \mu(z) \cdot f_z \tag{1.1}$$

where $f_{\overline{z}} = \overline{\partial} f = (f_x + i f_y)/2$, $f_z = \partial f = (f_x - i f_y)/2$, $z = x + iy$, and f_x and f_y are partial derivatives of f in x and y, correspondingly.

The function μ is called the **complex coefficient** and

$$K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{1.2}$$

the **maximal dilatation** or in short the **dilatation** of the equation (1.1). The Beltrami equation (1.1) is said to be **degenerate** if $ess\,sup\, K_\mu(z) = \infty$.

Use will be made also the **tangential dilatation** with respect to a point $z_0 \in \overline{D}$ which is defined by

$$(1.3) \qquad K_\mu^T(z, z_0) \ = \ \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1-|\mu(z)|^2}\ ,$$

cf. [An], [GMSV$_1$]–[GMSV$_2$], [Le], [RW], [RSY$_2$] and [RSY$_3$]. Note the following precise estimates

$$(1.4) \qquad \frac{1}{K_\mu(z)} \ \leq \ K_\mu^T(z, z_0) \ \leq \ K_\mu(z) \qquad \text{a.e.}$$

Thus, $K_\mu^T(z, z_0) \neq 0$ and ∞ a.e. if $K_\mu(z)$ is locally integrable in a domain D.

Recall that a mapping $f : D \to \mathbb{C}$ is **absolutely continuous on lines**, abbr. $f \in$**ACL**, if, for every closed rectangle R in D whose sides are parallel to the coordinate axes, $f|R$ is absolutely continuous on almost all line segments in R which are parallel to the sides of R. In particular, f is ACL if it belongs to the Sobolev class $W^{1,1}_{loc}$, see e.g. [Ma], p. 8. Note that, if $f \in$ ACL, then f has partial derivatives f_x and f_y a.e. Furthermore, every ACL homeomorphism is differentiable a.e., see e.g. [GL] or [LV], p. 128, or [Me] and [Tr], p. 331. For a sense-preserving ACL homeomorphism $f : D \to \mathbb{C}$, the Jacobian $J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2$ is nonnegative a.e., see [LV], p. 10. In this case, the **complex dilatation** of f is the ratio $\mu(z) = f_{\overline{z}}/f_z$, and $|\mu(z)| \leq 1$ a.e., and the **dilatation** of f is $K_\mu(z)$ from (1.2) and $K_\mu(z) \geq 1$ a.e. Here we set by definition $\mu(z) = 0$ and, correspondingly, $K_\mu(z) = 1$ if $f_z = 0$. The complex dilatation and the dilatation of f will be denoted by μ_f and K_f, respectively.

Recall also that, given a family of paths Γ in $\overline{\mathbb{C}}$, a Borel function $\rho : \overline{\mathbb{C}} \to [0, \infty]$ is called **admissible** for Γ, abbr. $\rho \in adm\,\Gamma$, if

$$(1.5) \qquad \int\limits_\gamma \rho(z)\,|dz| \ \geq \ 1$$

for each $\gamma \in \Gamma$. The **modulus** of Γ is defined by

$$(1.6) \qquad M(\Gamma) = \inf_{\rho\in adm\,\Gamma} \int\limits_{\mathbb{C}} \rho^2(z)\ dxdy\ .$$

Given a domain D and two sets E and F in $\overline{\mathbb{C}}$, $\Gamma(E, F, D)$ denotes the family of all paths $\gamma : [a, b] \to \overline{\mathbb{C}}$ which join E and F in D, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for $a < t < b$. We set $\Gamma(E, F) = \Gamma(E, F, \overline{\mathbb{C}})$ if $D = \overline{\mathbb{C}}$. A **ring domain**, or shortly a **ring** in $\overline{\mathbb{C}}$ is a domain R in $\overline{\mathbb{C}}$ whose complement consists of two components. Let R be a ring in $\overline{\mathbb{C}}$. If C_1 and C_2 are the components of $\overline{\mathbb{C}} \setminus R$, we write $R = R(C_1, C_2)$. It is known that $M(\Gamma(C_1, C_2, R)) = cap\ R(C_1, C_2)$, see e.g. [Ge$_1$]. Note also that $M(\Gamma(C_1, C_2, R)) = M(\Gamma(C_1, C_2))$, see e.g. Theorem 11.3 in [Va]. In what follows, we use the notations $B(z_0, r)$ and $C(z_0, r)$ for the

open disk and the circle, respectively, in $\mathbb{C}$ centered at $z_0 \in \mathbb{C}$ with the radius $r > 0$ and $A(z_0, r_1, r_2)$ for the ring $\{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$.

Motivated by the ring definition of quasiconformality in [Ge$_2$], we introduced in [RSY$_3$], cf. also [RSY$_2$], the following notion that localizes and extends the notion of a Q–homeomorphism, see e.g. [MRSY]. Let D be a domain in $\mathbb{C}$, $z_0 \in D$, $r_0 \leq \operatorname{dist}(z_0, \partial D)$ and $Q : B(z_0, r_0) \to [0, \infty]$ a measurable function. A homeomorphism $f : D \to \overline{\mathbb{C}}$ is called a **ring Q–homeomorphism at the point** $z_0 \in D$ if

$$M(\Gamma(fC_1, fC_2)) \ \leq \int\limits_{A} Q(z) \cdot \eta^2(|z - z_0|) \ dxdy \tag{1.7}$$

for every ring $A = A(z_0, r_1, r_2)$, $0 < r_1 < r_2 < r_0$, $C_i = C(z_0, r_i)$, $i = 1, 2$, and for every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r) \ dr \ = \ 1 \ . \tag{1.8}$$

This notion was first extended to the boundary points in [RSY$_5$]. More precisely, given a domain D in $\mathbb{C}$ and a measurable function $Q : D \to [0, \infty]$, we say that a homeomorphism $f : D \to \overline{\mathbb{C}}$ is a **ring $Q-$homeomorphism at a boundary point** z_0 of the domain D if

$$M(\Delta(fC_1, fC_2, fD)) \ \leq \int\limits_{A \cap D} Q(z) \cdot \eta^2(|z - z_0|) \ dxdy \tag{1.9}$$

for every ring $A = A(z_0, r_1, r_2)$ and every continua C_1 and C_2 in D which belong to the different components of the complement to the ring A in $\overline{\mathbb{C}}$, containing z_0 and ∞, correspondingly, and for every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ satisfying the condition (1.8).

An ACL homeomorphism $f_\mu : D \to \mathbb{C}$ is called a **ring solution** of the Beltrami equation (1.1) if f satisfies (1.1) a.e., $f^{-1} \in W^{1,2}_{loc}(f(D))$ and f is a ring Q–homeomorphism at every point $z_0 \in D$ with $Q(z) = K^T_\mu(z, z_0)$. If in addition f is a ring Q–homeomorphism at every boundary point $z_0 \in \partial D$ with $Q(z) = K^T_\mu(z, z_0)$, then f is called a **strong ring solution** of (1.1).

The inequality (1.9), which strong ring solutions satisfy, is an useful tool in deriving various, in particular, boundary properties of such solutions. The condition $f^{-1} \in W^{1,2}_{loc}$ given in the definition of a ring solution implies that a.e. point z is a **regular point** for the mapping f, i.e., f is differentiable at z and $J_f(z) \neq 0$, see [Po] and Theorem III.6.1 in [LV]. Note that the condition $K_\mu \in L^1_{loc}$ is necessary for a homeomorphic ACL solution f of (1.1) to have the property $g = f^{-1} \in W^{1,2}_{loc}$ because this property implies that

$$\int\limits_{C} K_\mu(z) \ dxdy \ \leq \ 4 \int\limits_{C} \frac{dxdy}{1 - |\mu(z)|^2} \ = \ 4 \int\limits_{f(C)} |\partial g|^2 \ dudv \ < \ \infty$$

for every compact set $C \subset D$, see e.g. Lemmas III.2.1, III.3.2 and Theorems III.3.1, III.6.1 in [LV], cf. also I.C(3) in [Ah]. Note also that every homeomorphic

ACL solution f of the Beltrami equation with $K_\mu \in L^1_{loc}$ belongs to the class $W^{1,1}_{loc}$ as in all our theorems. If in addition $K_\mu \in L^p_{loc}$, $p \in [1,\infty]$, then $f_\mu \in W^{1,s}_{loc}$ where $s = 2p/(1+p) \in [1,2]$. Indeed, if $f \in$ ACL, then f has partial derivatives f_x and f_y a.e. and, for a sense-preserving ACL homeomorphism $f : D \to \mathbb{C}$, the Jacobian $J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2$ is nonnegative a.e. and, moreover,

$$(1.10) \qquad |\,\overline{\partial} f| \ \le\ |\,\partial f| \ \le\ |\,\partial f| \ +\ |\,\overline{\partial} f| \ \le\ K_\mu^{1/2}(z) \cdot J_f^{1/2}(z) \qquad a.e.$$

Recall that if a homeomorphism $f : D \to \mathbb{C}$ has finite partial derivatives a.e., then

$$(1.11) \qquad \int\limits_B J_f(z)\ dxdy \ \le\ |f(B)|$$

for every Borel set $B \subseteq D$, see e.g. Lemma III.3.3 in [LV]. Consequently, applying successively the Hölder inequality and the inequality (1.11) to (1.10), we get that

$$(1.12) \qquad \|\partial f\|_s \ \le\ \|K_\mu\|_p^{1/2} \cdot |f(C)|^{1/2}$$

where $\|\cdot\|_s$ and $\|\cdot\|_p$ denote the L^s- and L^p-norms in a compact set $C \subset D$, respectively. In the classical case when $\|\mu\|_\infty < 1$, equivalently, when $K_\mu \in L^\infty$, every ACL homeomorphic solution f of the Beltrami equation (1.1) is in the class $W^{1,2}_{loc}$ together with its inverse mapping f^{-1}. In the case $\|\mu\|_\infty = 1$ and when $K_\mu \le Q \in$ BMO, again $f^{-1} \in W^{1,2}_{loc}$ and f belongs to $W^{1,s}_{loc}$ for all $1 \le s < 2$ but not necessarily to $W^{1,2}_{loc}$, see e.g. [RSY$_1$].

Olli Lehto considers in [Le] degenerate Beltrami equations in the special case where the **singular set** $S_\mu = \{z \in \mathbb{C} : \lim\limits_{\varepsilon\to 0} \ \|K_\mu\|_{L^\infty(B(z,\varepsilon))} \ =\ \infty\}$ of the complex coefficient μ in (1.1) is of measure zero and shows that, if for every $z_0 \in \mathbb{C}$, r_1 and $r_2 \in (0,\infty)$, $r_2 > r_1$, the following integral is positive and

$$(1.13) \qquad \int\limits_{r_1}^{r_2} \frac{dr}{r(1+q^T_{z_0}(r))} \ \to \infty \qquad \text{as } r_1 \to 0 \text{ or } r_2 \to \infty$$

where $q^T_{z_0}(r)$ is the average of $K^T_\mu(z,z_0)$ over $|z-z_0| = r$, then there exists a homeomorphism $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which is ACL in $\mathbb{C} \setminus S_\mu$ and satisfies (1.1) a.e.

Our extension and strengthening of Lehto's existence theorem for ring solutions was first published in the preprint [RSY$_2$], and then in the journal paper [RSY$_3$], see also [MRSY]. The most advanced version for the strong ring solutions is the following, see [RSY$_6$]:

1.14. Theorem. *Let D be a domain in $\mathbb{C}$ and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}(D)$. Suppose that*

$$(1.15) \qquad \int\limits_0^{\delta(z_0)} \frac{dr}{rq^T_{z_0}(r)} \ =\ \infty \qquad \forall\ z_0 \in D$$

where $\delta(z_0) < dist\,(z_0, \partial D)$ and $q^T_{z_0}(r)$ is the average of $K^T_\mu(z,z_0)$ over $|z-z_0| = r$. Then the Beltrami equation (1.1) has a strong ring solution.

Note that the situation where $S_\mu = D$ is possible here and that the condition (1.15) is a little weaker than the Lehto condition (1.13) because $q^T_{z_0}(r)$ can be arbitrarily close to 0, see (1.4). Note also that already in the work [MS] it was established the existence of homeomorphic solutions to (1.1) in the class $f_\mu \in W^{1,s}_{loc}$, $s = 2p/(1+p)$, under the condition (1.15) with $K_\mu \in L^p_{loc}$, $p > 1$, instead of $K^T_\mu(z, z_0)$ (the case $p = 1$ is covered thanking to a new convergence theorem in the recent paper [RSY$_5$], see also [RSY$_6$]). The Miklyukov-Suvorov result was again discovered in the paper [Ch] whose author thanks Professor F.W. Gehring but does not mention [MS]. Perhaps, the work [MS] remains unknown even for the leading experts in the west because we have found no reference to this work in the latest monograph in the Beltrami equation under the discussion of the Lehto condition, see Theorem 20.9.4 in [AIM].

Theorem 1.14, side by side with lemmas in Sections 2 and 3, is the the main base for deriving all theorems in Section 4 on existence of strong ring solutions for Beltrami equations with various integral conditions.

2 On some equivalent integral conditions

The main existence theorems for the Beltrami equations (1.1) are based on integral restrictions on the dilatations $K_\mu(z)$ and $K^T_\mu(z, z_0)$. Here we establish equivalence of a series of the corresponding integral conditions.

For this goal, we use the following notions of the inverse function for monotone functions. For every non-decreasing function $\Phi : [0,\infty] \to [0,\infty]$, the **inverse function** $\Phi^{-1} : [0,\infty] \to [0,\infty]$ can be well defined by setting

$$\Phi^{-1}(\tau) \ = \ \inf_{\Phi(t)\geq\tau} \ t \ . \tag{2.1}$$

Here inf is equal to ∞ if the set of $t \in [0,\infty]$ such that $\Phi(t) \geq \tau$ is empty. Note that the function Φ^{-1} is non-decreasing, too.

2.2. Remark. It is evident immediately by the definition that

$$\Phi^{-1}(\Phi(t)) \ \leq \ t \qquad \forall\, t \in [0,\infty] \tag{2.3}$$

with the equality in (2.3) except intervals of constancy of the function $\varphi(t)$.

Similarly, for every non-increasing function $\varphi : [0,\infty] \to [0,\infty]$, we set

$$\varphi^{-1}(\tau) \ = \ \inf_{\varphi(t)\leq\tau} \ t \ . \tag{2.4}$$

Again, here inf is equal to ∞ if the set of $t \in [0,\infty]$ such that $\varphi(t) \leq \tau$ is empty. Note that the function φ^{-1} is also non-increasing.

2.5. Lemma. *Let $\psi : [0,\infty] \to [0,\infty]$ be a sense–reversing homeomorphism and $\varphi : [0,\infty] \to [0,\infty]$ a monotone function. Then*

$$[\psi \circ \varphi]^{-1}(\tau) \ = \ \varphi^{-1} \circ \psi^{-1}(\tau) \qquad \forall \tau \in [0,\infty] \tag{2.6}$$

and

$$[\varphi\circ\psi]^{-1}(\tau) \ \leq \ \psi^{-1}\circ\varphi^{-1}(\tau) \qquad \forall\tau\in[0,\infty] \tag{2.7}$$

and, except a countable collection of $\tau\in[0,\infty]$,

$$[\varphi\circ\psi]^{-1}(\tau) \ = \ \psi^{-1}\circ\varphi^{-1}(\tau)\ . \tag{2.8}$$

The equality (2.8) holds for all $\tau\in[0,\infty]$ *iff the function* $\varphi:[0,\infty]\to[0,\infty]$ *is strictly monotone.*

2.9. Remark. If ψ is a sense–preserving homeomorphism, then (2.6) and (2.8) are obvious for every monotone function φ. Similar notations and statements also hold for other segments $[a,b]$, where a and $b\in[-\infty,+\infty]$, instead of the segment $[0,\infty]$.

Proof of Lemma 2.5. Let us first prove (2.6). If φ is non-increasing, then

$$[\psi\circ\varphi]^{-1}\,(\tau) = \inf_{\psi(\varphi(t))\geq\tau} t \ = \inf_{\varphi(t)\leq\psi^{-1}(\tau)} t \ = \ \varphi^{-1}\circ\psi^{-1}(\tau)\,.$$

Similarly, if φ is non-decreasing, then

$$[\psi\circ\varphi]^{-1}\,(\tau) = \inf_{\psi(\varphi(t))\leq\tau} t \ = \inf_{\varphi(t)\geq\psi^{-1}(\tau)} t \ = \ \varphi^{-1}\circ\psi^{-1}(\tau)\,.$$

Now, let us prove (2.7) and (2.8). If φ is non-increasing, then applying the substitution $\eta=\psi(t)$ we have

$$[\varphi\circ\psi]^{-1}\,(\tau) = \inf_{\varphi(\psi(t))\geq\tau} t \ = \inf_{\varphi(\eta)\geq\tau}\psi^{-1}(\eta) \ = \ \psi^{-1}\left(\sup_{\varphi(\eta)\geq\tau}\eta\right) \ \leq$$

$$\leq \ \psi^{-1}\left(\inf_{\varphi(\eta)\leq\tau}\eta\right) \ = \ \psi^{-1}\circ\varphi^{-1}\,(\tau)\ ,$$

i.e., (2.7) holds for all $\tau\in[0,\infty]$. It is evident that here the strict inequality is possible only for a countable collection of $\tau\in[0,\infty]$ because an interval of constancy of φ corresponds to every such τ. Hence (2.8) holds for all $\tau\in[0,\infty]$ if and only if φ is decreasing.

Similarly, if φ is non-decreasing, then

$$[\varphi\circ\psi]^{-1}\,(\tau) = \inf_{\varphi(\psi(t))\leq\tau} t \ = \inf_{\varphi(\eta)\leq\tau}\psi^{-1}(\eta) \ = \ \psi^{-1}\left(\sup_{\varphi(\eta)\leq\tau}\eta\right) \ \leq$$

$$\leq \ \psi^{-1}\left(\inf_{\varphi(\eta)\geq\tau}\eta\right) \ = \ \psi^{-1}\circ\varphi^{-1}\,(\tau)\ ,$$

i.e., (2.7) holds for all $\tau\in[0,\infty]$ and again the strict inequality is possible only for a countable collection of $\tau\in[0,\infty]$. In the case, (2.8) holds for all $\tau\in[0,\infty]$ if and only if φ is increasing.

2.10. Corollary. *In particular, if* $\varphi:[0,\infty]\to[0,\infty]$ *is a monotone function and* $\psi=j$ *where* $j(t)=1/t$, *then* $j^{-1}=j$ *and*

$$[j\circ\varphi]^{-1}(\tau) \ = \ \varphi^{-1}\circ j(\tau) \qquad \forall\tau\in[0,\infty] \tag{2.11}$$

i.e.,

(2.12) $$\varphi^{-1}(\tau) \ = \ \Phi^{-1}(1/\tau) \qquad \forall \tau \in [0,\infty]$$

where $\Phi = 1/\varphi$,

(2.13) $$[\varphi \circ j]^{-1}(\tau) \ \leq \ j \circ \varphi^{-1}(\tau) \qquad \forall \tau \in [0,\infty]$$

i.e., the inverse function of $\varphi(1/t)$ *is dominated by* $1/\varphi^{-1}$, *and except a countable collection of* $\tau \in [0,\infty]$

(2.14) $$[\varphi \circ j]^{-1}(\tau) \ = \ j \circ \varphi^{-1}(\tau) \ .$$

$1/\varphi^{-1}$ *is the inverse function of* $\varphi(1/t)$ *if and only if the function* φ *is strictly monotone.*

Further, the integral in (2.18) is understood as the Lebesgue–Stieltjes integral and the integrals in (2.17) and (2.19)–(2.22) as the ordinary Lebesgue integrals. In (2.17) and (2.18) we complete the definition of integrals by ∞ if $\Phi(t) = \infty$, correspondingly, $H(t) = \infty$, for all $t \geq T \in [0,\infty)$.

2.15. Theorem. *Let* $\Phi : [0,\infty] \to [0,\infty]$ *be a non-decreasing function and set*

(2.16) $$H(t) \ = \ \log \Phi(t) \ .$$

Then the equality

(2.17) $$\int\limits_{\Delta}^{\infty} H'(t) \, \frac{dt}{t} \ = \ \infty$$

implies the equality

(2.18) $$\int\limits_{\Delta}^{\infty} \frac{dH(t)}{t} \ = \ \infty$$

and (2.18) is equivalent to

(2.19) $$\int\limits_{\Delta}^{\infty} H(t) \, \frac{dt}{t^2} \ = \ \infty$$

for some $\Delta > 0$, *and (2.19) is equivalent to every of the equalities:*

(2.20) $$\int\limits_{0}^{\delta} H\left(\frac{1}{t}\right) \, dt \ = \ \infty$$

for some $\delta > 0$,

(2.21) $$\int\limits_{\Delta_*}^{\infty} \frac{d\eta}{H^{-1}(\eta)} \ = \ \infty$$

for some $\Delta_* > H(+0)$,

(2.22) $$\int\limits_{\delta_*}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} \ = \ \infty$$

for some $\delta_* > \Phi(+0)$.

Moreover, (2.17) is equivalent to (2.18) and hence (2.17)–(2.22) are equivalent each to other if Φ *is in addition absolutely continuous. In particular, all the conditions (2.17)–(2.22) are equivalent if* Φ *is convex and non–decreasing.*

2.23. Remark. It is necessary to give one more explanation. From the right hand sides in the conditions (2.17)–(2.22) we have in mind $+\infty$. If $\Phi(t) = 0$ for $t \in [0, t_*]$, then $H(t) = -\infty$ for $t \in [0, t_*]$ and we complete the definition $H'(t) = 0$ for $t \in [0, t_*]$. Note, the conditions (2.18) and (2.19) exclude that t_* belongs to the interval of integrability because in the contrary case the left hand sides in (2.18) and (2.19) are either equal to $-\infty$ or indeterminate. Hence we may assume in (2.17)–(2.20) that $\Delta > t_0$ where $t_0 := \sup\limits_{\Phi(t)=0} t$, $t_0 = 0$ if $\Phi(0) > 0$, and $\delta < 1/t_0$, correspondingly.

Proof. The equality (2.17) implies (2.18) because except the mentioned special case

$$\int\limits_{\Delta}^{T} d\Psi(t) \ \geq \ \int\limits_{\Delta}^{T} \Psi'(t)\ dt \qquad \forall\ T \in (\Delta,\ \infty)$$

where

$$\Psi(t) := \int\limits_{\Delta}^{t} \frac{d\,H(\tau)}{\tau}\ , \quad \Psi'(t) \ = \ \frac{H'(t)}{t}\ ,$$

see e.g. Theorem 7.4 of Chapter IV in [Sa], p. 119, and hence

$$\int\limits_{\Delta}^{T} \frac{d\,H(t)}{t} \ \geq \ \int\limits_{\Delta}^{T} H'(t)\ \frac{dt}{t} \qquad \forall\ T \in (\Delta,\ \infty)$$

The equality (2.18) is equivalent to (2.19) by integration by parts, see e.g. Theorem III.14.1 in [Sa], p. 102. Indeed, again except the mentioned special case, through integration by parts we have

$$\int\limits_{\Delta}^{T} \frac{d\,H(t)}{t} \ - \ \int\limits_{\Delta}^{T} H(t)\ \frac{dt}{t^2} \ = \ \frac{H(T+0)}{T} \ - \ \frac{H(\Delta-0)}{\Delta} \qquad \forall\ T \in (\Delta,\ \infty)$$

and, if

$$\liminf_{t\to\infty} \ \frac{H(t)}{t} \ < \infty\ ,$$

then the equivalence of (2.18) and (2.19) is obvious. If

$$\lim_{t\to\infty} \ \frac{H(t)}{t} \ = \infty\ ,$$

then (2.19) obviously holds, $\frac{H(t)}{t} \geq 1$ for $t > t_0$ and

$$\int\limits_{t_0}^{T} \frac{d\,H(t)}{t} = \int\limits_{t_0}^{T} \frac{H(t)}{t}\ \frac{d\,H(t)}{H(t)} \geq \log\ \frac{H(T)}{H(t_0)} = \log\ \frac{H(T)}{T} + \log\ \frac{T}{H(t_0)} \to \infty$$

as $T \to \infty$, i.e. (2.18) holds, too.

Now, (2.19) is equivalent to (2.20) by the change of variables $t \to 1/t$.

Next, (2.20) is equivalent to (2.21) because by the geometric sense of integrals as areas under graphs of the corresponding integrands

$$\int_0^\delta \Psi(t)\ dt \ = \ \int_{\Psi(\delta)}^\infty \Psi^{-1}(\eta)\ d\eta \ + \ \delta \cdot \Psi(\delta)$$

where $\Psi(t) = H(1/t)$, and because by Corollary 2.10 the inverse function for $H\,(1/t)$ coincides with $1/H^{-1}$ at all points except a countable collection.

Further, set $\psi(\xi) = \log \xi$. Then $H = \psi \circ \Phi$ and by Lemma 2.5 and Remark 2.9 $H^{-1} = \Phi^{-1} \circ \psi^{-1}$, i.e., $H^{-1}(\eta) = \Phi^{-1}(e^\eta)$, and by the substitutions $\tau = e^\eta$, $\eta = \log\ \tau$ we have the equivalence of (2.21) and (2.22).

Finally, (2.17) and (2.18) are equivalent if Φ is absolutely continuous, see e.g. Theorem IV.7.4 in [Sa] p. 119.

3 Connection with the Lehto condition

In this section we establish useful connection of the conditions of the Lehto type (1.15) with one of the integral conditions from the last section.

Recall that a function $\psi : [0, \infty] \to [0, \infty]$ is called **convex** if $\psi(\lambda t_1 + (1-\lambda)t_2) \le \lambda\psi(t_1) + (1-\lambda)\psi(t_2)$ for all t_1 and $t_2 \in [0, \infty]$ and $\lambda \in [0, 1]$.

In what follows, $\mathbb{D}$ denotes the unit disk in the complex plane $\mathbb{C}$,

$$\mathbb{D} \ = \ \{\ z \in \mathbb{C} :\ |z| \ < \ 1\ \}\ . \tag{3.1}$$

3.2. Lemma. *Let $Q : \mathbb{D} \to [0, \infty]$ be a measurable function and let $\Phi : [0, \infty] \to [0, \infty]$ be a non-decreasing convex function. Then*

$$\int_0^1 \frac{dr}{rq(r)} \ \ge \ \frac{1}{2} \int_N^\infty \frac{d\tau}{\tau \Phi^{-1}(\tau)} \tag{3.3}$$

where $q(r)$ is the average of the function $Q(z)$ over the circle $|z| = r$ and

$$N \ = \ \int_{\mathbb{D}} \Phi(Q(z))\ dxdy\ . \tag{3.4}$$

Proof. Note that the result is obvious if $N = \infty$. Hence we assume further that $N < \infty$. Consequently, we may also assume that $\Phi(t) < \infty$ for all $t \in [0, \infty)$ because in the contrary case $Q \in L^\infty(\mathbb{D})$ and then the left hand side in (3.3) is equal to ∞. Moreover, we may assume that $\Phi(t)$ is not constant (because in the contrary case $\Phi^{-1}(\tau) \equiv \infty$ for all $\tau > \tau_0$ and hence the right hand side in (3.3)

is equal to 0), $\Phi(t)$ is (strictly) increasing, convex and continuous in a segment $[t_*, \infty]$ for some $t_* \in [0, \infty)$ and

$$\Phi(t) \;\equiv\; \tau_0 \;=\; \Phi(0) \qquad \forall\; t \;\in [0, t_*]\;. \tag{3.5}$$

Next, setting

$$H(t) := \;\log\; \Phi(t)\;, \tag{3.6}$$

we see by Proposition 2.5 and Remark 2.9 that

$$H^{-1}(\eta) \;=\; \Phi^{-1}(e^{\eta})\;, \quad \Phi^{-1}(\tau) \;=\; H^{-1}(\log\;\tau)\;. \tag{3.7}$$

Thus, we obtain that

$$q(r) = H^{-1}\left(\log \frac{h(r)}{r^2}\right) = H^{-1}\left(2 \log \frac{1}{r} + \log\; h(r)\right) \qquad \forall\; r \;\in R_* \tag{3.8}$$

where $h(r) := \; r^2 \Phi(q(r))$ and $R_* \;=\; \{r \in (0,1) :\; q(r) \;>\; t_*\}$. Then also

$$q(e^{-s}) \;=\; H^{-1}\left(2s \;+\; \log\; h(e^{-s})\right) \qquad \forall\; s \;\in S_* \tag{3.9}$$

where $S_* \;=\; \{s \in (0, \infty) :\; q(e^{-s}) \;>\; t_*\}$.

Now, by the Jensen inequality

$$\begin{aligned} \int\limits_0^{\infty} h(e^{-s})\; ds \;&=\; \int\limits_0^1 h(r)\; \frac{dr}{r} \;=\; \int\limits_0^1 \Phi(q(r))\; rdr \\ &\leq\; \int\limits_0^1 \left(\frac{1}{2\pi} \int\limits_0^{2\pi} \Phi(Q(re^{i\vartheta}))\; d\vartheta \right) rdr \;=\; \frac{N}{2\pi} \end{aligned} \tag{3.10}$$

and then

$$|T| \;=\; \int\limits_T ds \;\leq\; \frac{1}{2} \tag{3.11}$$

where $T \;=\; \{\; s \in (0, \infty) : \quad h(e^{-s}) \;>\; N/\pi\}$. Let us show that

$$q(e^{-s}) \;\leq\; H^{-1}\left(2s \;+\; \log\; \frac{N}{\pi}\right) \qquad \forall\; s \in (0, \infty) \setminus T_* \tag{3.12}$$

where $T_* \;=\; T \cap S_*$. Note that $(0, \infty) \setminus T_* \;=\; [(0, \infty) \setminus S_*] \cup [(0, \infty) \setminus T] \;=\; [(0, \infty) \setminus S_*] \cup [S_* \setminus T]$. The inequality (3.12) holds for $s \in S_* \setminus T$ by (3.9) because H^{-1} is a non-decreasing function. Note also that by (3.5)

$$e^{2s} \frac{N}{\pi} = e^{2s} \fint_{\mathbb{D}} \Phi(Q(z))\; dxdy > \Phi(0) = \tau_0 \qquad \forall\; s \in (0, \infty)\;. \tag{3.13}$$

Hence, since the function Φ^{-1} is non-decreasing and $\Phi^{-1}(\tau_0 + 0) = t_*$, we have by (3.7) that

$$t_* < \Phi^{-1}\left(\frac{N}{\pi}\; e^{2s}\right) = H^{-1}\left(2s \;+\; \log\; \frac{N}{\pi}\right) \qquad \forall\; s \in (0, \infty)\;. \tag{3.14}$$

Consequently, (3.12) holds for $s \in (0,\infty) \setminus S_*$, too. Thus, (3.12) is true.

Since H^{-1} is non–decreasing, we have by (3.11) and (3.12) that

$$\int\limits_0^1 \frac{dr}{rq(r)} = \int\limits_0^\infty \frac{ds}{q(e^{-s})} \geq \int\limits_{(0,\infty)\setminus T_*} \frac{ds}{H^{-1}(2s+\Delta)} \geq \tag{3.15}$$

$$\geq \int\limits_{|T_*|}^\infty \frac{ds}{H^{-1}(2s+\Delta)} \geq \int\limits_{\frac{1}{2}}^\infty \frac{ds}{H^{-1}(2s+\Delta)} = \frac{1}{2}\int\limits_{1+\Delta}^\infty \frac{d\eta}{H^{-1}(\eta)}$$

where $\Delta = \log N/\pi$. Note that $1+\Delta = \log\ N + \log\ e/\pi\ < \log\ N$. Thus,

$$\int\limits_0^1 \frac{dr}{rq(r)} \geq \frac{1}{2}\int\limits_{\log N}^\infty \frac{d\eta}{H^{-1}(\eta)} \tag{3.16}$$

and, after the replacement $\eta = \log\ \tau$, we obtain (3.3).

3.17. Theorem. *Let $Q : \mathbb{D} \to [0,\infty]$ be a measurable function such that*

$$\int\limits_{\mathbb{D}} \Phi(Q(z))\ dxdy\ <\ \infty \tag{3.18}$$

where $\Phi : [0,\infty] \to [0,\infty]$ is a non-decreasing convex function such that

$$\int\limits_{\delta_0}^\infty \frac{d\tau}{\tau\Phi^{-1}(\tau)} = \infty \tag{3.19}$$

for some $\delta_0\ >\ \tau_0 :=\ \Phi(0)$. Then

$$\int\limits_0^1 \frac{dr}{rq(r)} = \infty \tag{3.20}$$

where $q(r)$ is the average of the function $Q(z)$ over the circle $|z| = r$.

3.21. Remark. Note that (3.19) implies that

$$\int\limits_{\delta}^\infty \frac{d\tau}{\tau\Phi^{-1}(\tau)} = \infty \tag{3.22}$$

for every $\delta\ \in\ [0,\infty)$ but (3.22) for some $\delta \in [0,\infty)$, generally speaking, does not imply (3.19). Indeed, for $\delta \in [0,\delta_0)$, (3.19) evidently implies (3.22) and, for $\delta \in (\delta_0,\infty)$, we have that

$$0 < \int\limits_{\delta_0}^{\delta} \frac{d\tau}{\tau\Phi^{-1}(\tau)} \leq \frac{1}{\Phi^{-1}(\delta_0)}\ \log\ \frac{\delta}{\delta_0} < \infty \tag{3.23}$$

because Φ^{-1} is non-decreasing and $\Phi^{-1}(\delta_0) > 0$. Moreover, by the definition of the inverse function $\Phi^{-1}(\tau) \equiv 0$ for all $\tau \in [0, \tau_0]$, $\tau_0 = \Phi(0)$, and hence (3.22) for $\delta \in [0, \tau_0)$, generally speaking, does not imply (3.19). If $\tau_0 > 0$, then

$$\int\limits_{\delta}^{\tau_0} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \qquad \forall\ \delta \in [0, \tau_0) \tag{3.24}$$

However, (3.24) gives no information on the function $Q(z)$ itself and, consequently, (3.22) for $\delta < \Phi(0)$ cannot imply (3.20) at all.

By (3.22) the proof of Theorem 3.17 is reduced to Lemma 3.2. Combining Theorems 2.15 and 3.17 we also obtain the following conclusion.

3.25. Corollary. *If $\Phi : [0, \infty] \to [0, \infty]$ is a non-decreasing convex function and Q satisfies the condition (3.18), then every of the conditions (2.17)–(2.22) implies (3.20).*

4 Sufficient conditions for solvability

The following existence theorem is obtained immediately from Theorems 1.14 and 3.17.

4.1. Theorem. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. Suppose that every point $z_0 \in D$ has a neighborhood U_{z_0} where*

$$\int\limits_{U_{z_0}} \Phi_{z_0}(K^T_\mu(z, z_0))\, dxdy < \infty \tag{4.2}$$

for a non-decreasing convex function $\Phi_{z_0} : [0, \infty) \to [0, \infty]$ such that

$$\int\limits_{\Delta(z_0)}^{\infty} \frac{d\tau}{\tau \Phi^{-1}_{z_0}(\tau)} = \infty \tag{4.3}$$

for some $\Delta(z_0) > \Phi_{z_0}(0)$. Then the Beltrami equation (1.1) has a strong ring solution.

Proof. Let the closure of a disk $B(z_0, \rho)$ belong to the neighborhood U_{z_0}. Then we obtain by Theorem 3.17 applied to $Q(\zeta) = K^T_\mu(z_0 + \rho\,\zeta, z_0)$, $\zeta \in \mathbb{D}$, and $\Phi(t) = \Phi_{z_0}(t)$ that

$$\int\limits_{0}^{\rho} \frac{dr}{r q^T_{z_0}(r)} = \infty \tag{4.4}$$

where $q^T_{z_0}(r)$ is the mean value of $K^T_\mu(z, z_0)$ over the circle $|z - z_0| = r$. Thus, we have the desired conclusion by Theorem 1.14.

4.5. Remark. Note that the additional condition $\Delta(z_0) > \Phi_{z_0}(0)$ is essential, see also Remark 3.21. In fact, it is important only degree of convergence

$\Phi_{z_0}^{-1}(\tau) \to \infty$ as $\tau \to \infty$ or, the same, degree of convergence $\Phi_{z_0}(t) \to \infty$ as $t \to \infty$.

4.6. Corollary. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. Suppose that*

$$\int\limits_D \Phi(K_\mu(z))\, dxdy \ < \ \infty \tag{4.7}$$

for a non-decreasing convex function $\Phi : [0, \infty] \to [0, \infty]$ such that

$$\int\limits_\Delta^\infty \frac{d\tau}{\tau \Phi^{-1}(\tau)} \ = \ \infty \tag{4.8}$$

for some $\Delta > \Phi(0)$. Then the Beltrami equation (1.1) has a strong ring solution.

4.9. Remark. Applying transformations $\alpha \cdot \Phi + \beta$ with $\alpha > 0$ and $\beta \in \mathbb{R}$, we may assume without loss of generality that $\Phi(t) = \Phi(1) = 1$ for all $t \in [0, 1]$ and, thus, $\Phi(0) = \Phi(1) = 1$ in Theorem 4.1 and its corollaries further.

Many other criteria of the existence of strong ring solutions for the Beltrami equation (1.1) formulated below follow from Theorems 2.15 and 4.1.

4.10. Corollary. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. If the condition (4.2) holds at every point $z_0 \in D$ with a non-decreasing convex function $\Phi_{z_0} : [0, \infty) \to [0, \infty)$ such that*

$$\int\limits_{\Delta(z_0)}^\infty \log \Phi_{z_0}(t)\, \frac{dt}{t^2} \ = \ \infty \tag{4.11}$$

for some $\Delta(z_0) > 0$, then (1.1) has a strong ring solution.

4.12. Corollary. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. If the condition (4.2) holds at every point $z_0 \in D$ for a continuous non–decreasing convex function $\Phi_{z_0} : [0, \infty) \to [0, \infty)$ such that*

$$\int\limits_{\Delta(z_0)}^\infty (\log \Phi_{z_0}(t))'\, \frac{dt}{t} \ = \ \infty \tag{4.13}$$

for some $\Delta(z_0) > 0$, then (1.1) has a strong ring solution.

4.14. Corollary. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. If the condition (4.2) holds at every point $z_0 \in D$ for $\Phi_{z_0} = \exp H_{z_0}$ where H_{z_0} is non–constant, non–decreasing and convex, then (1.1) has a strong ring solution.*

4.15. Corollary. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. If the condition (4.2) holds at every point $z_0 \in D$ for*

$\Phi_{z_0} = \exp H_{z_0}$ with a twice continuously differentiable increasing function H_{z_0} such that

(4.16) $$H''_{z_0}(t)) \ \geq \ 0 \qquad \forall\, t \ \geq \ t(z_0) \ \in \ [0,\infty)\ ,$$

then (1.1) has a strong ring solution.

4.17. Theorem. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$ such that*

(4.18) $$\int\limits_D \Phi(K_\mu(z))\ dxdy \ < \ \infty$$

where $\Phi : [0,\infty) \to [0,\infty]$ is non-decreasing and convex such that

(4.19) $$\int\limits_\Delta^\infty \log\Phi(t)\ \frac{dt}{t^2} \ = \ \infty$$

for some $\Delta > 0$. Then the Beltrami equation (1.1) has a strong ring solution.

4.20. Corollary. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. If the condition (4.18) holds with a non–decreasing convex function $\Phi : [0,\infty) \to [0,\infty)$ such that*

(4.21) $$\int\limits_{t_0}^\infty (\log\Phi(t))'\ \frac{dt}{t} \ = \ \infty$$

for some $t_0 > 0$, then (1.1) has a strong ring solution.

4.22. Corollary. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. If the condition (4.18) holds for $\Phi = e^H$ where H is non–constant, non–decreasing and convex, then (1.1) has a strong ring solution.*

4.23. Corollary. *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. If the condition (4.18) holds for $\Phi = e^H$ where H is twice continuously differentiable, increasing and*

(4.24) $$H''(t) \ \geq \ 0 \qquad \forall\, t \ \geq \ t_0 \ \in \ [1,\infty)\ ,$$

then (1.1) has a strong ring solution.

Note that among twice continuously differentiable functions, the condition (4.24) is equivalent to the convexity of $H(t)$, $t \geq t_0$, cf. Corollary 4.22. Of course, the convexity of $H(t)$ implies the convexity of $\Phi(t) = e^{H(t)}$, $t \geq t_0$, because the function $\exp x$ is convex. However, in general, the convexity of Φ does not imply the convexity of $H(t) = \log\Phi(t)$ and it is known that the convexity of $\Phi(t)$ in (4.18) is not sufficient for the existence of ACL homeomorphic solutions of the Beltrami equation. There exist examples of the complex coefficients μ such that $K_\mu \in L^p$ with an arbitrarily large $p \geq 1$ for which the Beltrami equation (1.1) has no ACL homeomorphic solutions, see e.g. [RSY_1].

4.25. Remark. Theorem 1.14 is extended by us to the case where $\infty \in D \subset \overline{\mathbb{C}}$ in the standard way by replacing (1.15) to the following condition at ∞

$$\int\limits_{\delta}^{\infty} \frac{dr}{rq(r)} \;=\; \infty \tag{4.26}$$

where $\delta > 0$ and

$$q(r) \;=\; \frac{1}{2\pi} \int\limits_{0}^{2\pi} \frac{|1 - e^{-2i\vartheta}\mu(re^{i\vartheta})|^2}{1 - |\mu(re^{i\vartheta})|^2} \; d\vartheta \; . \tag{4.27}$$

In this case, there exists a homeomorphic $W^{1,1}_{loc}$ solution f of (1.1) in D with $f(\infty) = \infty$ and $f^{-1} \in W^{1,2}_{loc}$. Here $f \in W^{1,1}_{loc}$ in D means that $f \in W^{1,1}_{loc}$ in $D \setminus \{\infty\}$ and that $f^*(z) = 1/\overline{f(1/\overline{z})}$ belongs to $W^{1,1}$ in a neighborhood of 0. The statement $f^{-1} \in W^{1,2}_{loc}$ has a similar meaning.

Similarly, the integral condition (4.2) is replaced at ∞ by the following condition

$$\int\limits_{|z|>\delta} \Phi_\infty(K^T_\mu(z,\infty)) \, \frac{dxdy}{|z|^4} \;<\; \infty \tag{4.28}$$

where $\delta > 0$, Φ_∞ satisfies the conditions of either Theorem 4.1 or equivalent conditions from Theorem 2.15 and

$$K^T_\mu(z,\infty) \;=\; \frac{\left|1 - \frac{\overline{z}}{z}\mu(z)\right|^2}{1 - |\mu(z)|^2} \; . \tag{4.29}$$

We may assume in all the above theorems that the functions $\Phi_{z_0}(t)$ and $\Phi(t)$ are not convex on the whole segments $[0,\infty]$ and $[1,\infty]$, respectively, but only on a segment $[T,\infty]$ for some $T \in (1,\infty)$. Indeed, every non-decreasing function $\Phi : [1,\infty] \to [0,\infty]$ which is convex on a segment $[T,\infty]$, $T \in (0,\infty)$, can be replaced by a non-decreasing convex function $\Phi_T : [0,\infty] \to [0,\infty]$ in the following way. We set $\Phi_T(t) \equiv 0$ for all $t \in [0,T]$, $\Phi(t) = \varphi(t)$, $t \in [T,T_*]$, and $\Phi_T \equiv \Phi(t)$, $t \in [T_*,\infty]$, where $\tau = \varphi(t)$ is the line passing through the point $(0,T)$ and touching upon the graph of the function $\tau = \Phi(t)$ at a point $(T_*, \Phi(T_*))$, $T_* \geq T$. For such a function we have by the construction that $\Phi_T(t) \leq \Phi(t)$ for all $t \in [1,\infty]$ and $\Phi_T(t) = \Phi(t)$ for all $t \geq T_*$.

5 Necessary conditions for solvability

The main idea for the proof of the following statement under smooth increasing functions Φ with the additional condition that $t(\log \Phi)' \geq 1$ is due to Iwaniec and Martin, see Theorem 3.1 in [IM$_2$], cf. also Theorem 11.2.1 in [IM$_1$] and Theorem 20.3.1 in [AIM]. We obtain the same conclusion in Theorem 5.1 and Lemma 5.4 further without these smooth conditions. Moreover, by Theorem 2.15 the same conclusion concerns to all conditions (2.18)–(2.22).

5.1. Theorem. *Let $\Phi : [0,\infty] \to [0,\infty]$ be such a non-decreasing convex function that, for every measurable function $\mu : \mathbb{D} \to \mathbb{D}$ satisfying the condition*

$$\int\limits_{\mathbb{D}} \Phi(K_\mu(z))\ dxdy\ <\ \infty\ , \tag{5.2}$$

the Beltrami equation (1.1) has a homeomorphic ACL solution. Then there is $\delta > 0$ such that

$$\int\limits_{\delta}^{\infty} \log \Phi(t)\ \frac{dt}{t^2}\ =\ \infty\ . \tag{5.3}$$

It is evident that the function $\Phi(t)$ in Theorem 5.1 is not constant on $[0,\infty)$ because in the contrary case we would have no real restrictions for K_μ from (5.2) except $\Phi(t) \equiv \infty$ when the class of such μ is empty. Moreover, by the well–known criterion of convexity, see e.g. Proposition 5 in I.4.3 of [Bou], the inclination $[\Phi(t) - \Phi(0)]/t$ is nondecreasing. Hence the proof of Theorem 5.1 is reduced to the following statement.

5.4. Lemma. *Let a function $\Phi : [0,\infty] \to [0,\infty]$ be non-decreasing and*

$$\Phi(t)\ \geq\ C\cdot t \qquad \forall\ t\ \geq\ T \tag{5.5}$$

for some $C > 0$ and $T \in (1,\infty)$. If the Beltrami equations (1.1) have ACL homeomorphic solutions for all measurable functions $\mu : \mathbb{D} \to \mathbb{D}$ satisfying the condition (5.2), then (5.3) holds for some $\delta > 0$.

5.6. Remark. Note that the Iwaniec–Martin condition $t(\log\Phi)' \geq 1$ implies the condition (5.5) with $C = \Phi(T)/T$. Note also that if we take further in the construction of Lemma 6.12 $\beta_{n+1} = \varepsilon_n \to 0$ as $n \to \infty$, $\alpha_{n+1} = b_n e^{b_n\gamma_n} - \varepsilon_n\gamma_n$ and $\gamma^*_{n+1} = b_n e^{b_n\gamma_n}/\varepsilon_n$, then $\Phi(\gamma^*_{n+1})/\gamma^*_{n+1} \leq 2\varepsilon_n$ and we obtain examples of absolutely continuous increasing functions Φ with $\Phi(t) \to \infty$ as $t \to \infty$ satisfying the condition (5.3) and simultaneously

$$\liminf_{t\to\infty}\ \frac{\Phi(t)}{t}\ =\ 0\ . \tag{5.7}$$

Thus, conditions of the type (5.5) are independent on the conditions (2.17)–(2.22).

Proof of Lemma 5.4. Let us assume that the condition (5.3) does not hold for any $\delta > 0$. Set $t_0 = \sup\limits_{\Phi(t)=0} t$, $t_0 = 0$ if $\Phi(t) > 0$ for all $t \in [0,\infty]$. Then for all $\delta > t_0$

$$\int\limits_{\delta}^{\infty} \log \Phi(t)\ \frac{dt}{t^2}\ <\ \infty\ . \tag{5.8}$$

With no loss of generality, applying the linear transformation $\alpha\Phi + \beta$ with $\alpha = 1/C$ and $\beta = T$, we may assume by (5.5) that

$$\Phi(t)\ \geq\ t \qquad\quad \forall\ t \in [0,\infty)\ . \tag{5.9}$$

Of course, we may also assume that $\Phi(t) = t$ for all $t \in [0, 1)$ because the values of Φ in $[0, 1)$ give no information on K_μ in (5.2). Finally, by (5.8) we have that $\Phi(t) < \infty$ for every $t \in [0, \infty)$.

Now, note that the function $\Psi(t) := t\Phi(t)$ is strictly increasing, $\Psi(1) = \Phi(1)$ and $\Psi(t) \to \infty$ as $t \to \infty$, $\Psi(t) < \infty$ for every $t \in [0, \infty)$. Hence the functional equation

$$\Psi(K(r)) = \left(\frac{\gamma}{r}\right)^2 \qquad \forall\, r \in (0, 1]\,, \tag{5.10}$$

where $\gamma = \Phi^{1/2}(1) \geq 1$, is well solvable with $K(1) = 1$ and a continuous non-increasing function $K : (0, 1] \to [1, \infty)$ such that $K(r) \to \infty$ as $r \to 0$. Taking the logarithm in (5.10), we have that

$$2 \log\, r \;+\; \log\, K(r) \;+\; \log\, \Phi(K(r)) \;=\; 2\, log\, \gamma$$

and by (5.9) we obtain that

$$\log\, r \;+\; \log\, K(r) \;\leq\; log\, \gamma\,,$$

i.e.,

$$K(r) \;\leq\; \frac{\gamma}{r}\,. \tag{5.11}$$

Then by (5.10)

$$\Phi(K(r)) \;\geq\; \frac{\gamma}{r}$$

and hence by (2.3)

$$K(r) \;\geq\; \Phi^{-1}\left(\frac{\gamma}{r}\right)\,.$$

Thus,

$$I(t) := \int\limits_0^t \frac{dr}{rK(r)} \;\leq\; \int\limits_0^t \frac{dr}{r\Phi^{-1}\left(\frac{\gamma}{r}\right)} \;=\; \int\limits_{\frac{\gamma}{t}}^{\infty} \frac{d\tau}{\tau\Phi^{-1}(\tau)}, \qquad t \in (0, 1]\,,$$

where $\gamma/t \geq \gamma \geq 1 > \Phi(+0) = 0$. Hence by the condition (5.8) and Proposition 2.15

$$I(t) \;\leq\; I(1) \;=\; \int\limits_0^1 \frac{dr}{rK(r)} \;<\; \infty\,. \tag{5.12}$$

Next, consider the mapping

$$f(z) \;=\; \frac{z}{|z|}\,\rho(|z|)$$

where $\rho(t) = e^{I(t)}$. Note that $f \in C^1(\mathbb{D}\backslash\{0\})$ and hence f is locally quasiconformal in the punctured unit disk $\mathbb{D}\backslash\{0\}$ by the continuity of the function $K(r)$, $r \in (0, 1)$, see also (5.11). Let us calculate its complex dilatation. Set $z = re^{i\vartheta}$. Then

$$\frac{\partial f}{\partial r} \;=\; \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r} \;+\; \frac{\partial f}{\partial \overline{z}} \cdot \frac{\partial \overline{z}}{\partial r} \;=\; e^{i\vartheta} \cdot \frac{\partial f}{\partial z} \;+\; e^{-i\vartheta} \cdot \frac{\partial f}{\partial \overline{z}}$$

and
$$\frac{\partial f}{\partial \vartheta} = \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial \vartheta} + \frac{\partial f}{\partial \overline{z}}\cdot\frac{\partial \overline{z}}{\partial \vartheta} = ire^{i\vartheta}\cdot\frac{\partial f}{\partial z} - ire^{-i\vartheta}\cdot\frac{\partial f}{\partial \overline{z}} .$$
In other words,
$$\frac{\partial f}{\partial z} = \frac{e^{-i\vartheta}}{2}\left(\frac{\partial f}{\partial r} + \frac{1}{ir}\cdot\frac{\partial f}{\partial \vartheta}\right) \tag{5.13}$$
and
$$\frac{\partial f}{\partial \overline{z}} = \frac{e^{i\vartheta}}{2}\left(\frac{\partial f}{\partial r} - \frac{1}{ir}\cdot\frac{\partial f}{\partial \vartheta}\right) \tag{5.14}$$

Thus, we have that
$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\rho(r)}{rK(r)} + \frac{\rho(r)}{r}\right) = \frac{\rho(r)}{2r}\cdot\frac{1+K(r)}{K(r)}$$
and
$$\frac{\partial f}{\partial \overline{z}} = \frac{e^{2i\vartheta}}{2}\left(\frac{\rho(r)}{rK(r)} - \frac{\rho(r)}{r}\right) = e^{2i\vartheta}\cdot\frac{\rho(r)}{2r}\cdot\frac{1-K(r)}{K(r)}$$
i.e.
$$\mu(z) = e^{2i\vartheta}\cdot\frac{1-K(r)}{1+K(r)} = -\frac{z}{\overline{z}}\cdot\frac{K(|z|)-1}{K(|z|)+1} .$$
Consequently,
$$K_\mu(z) = K(|z|) \tag{5.15}$$
and by (5.10)
$$\int\limits_{\mathbb{D}} \Phi(K_\mu(z))\, dxdy = 2\pi \int\limits_0^1 \Phi(K(r))\, r\, dr \le 2\pi\gamma^2 I(1) < \infty .$$
However,
$$\lim_{z\to 0} |f(z)| = \lim_{t\to 0} \rho(t) = e^{I(0)} = 1 ,$$
i.e. f maps the punctured disk $\mathbb{D}\setminus\{0\}$ onto the ring $1<|\zeta|<R=e^{I(1)}$.

Let us assume that there is a homeomorphic ACL solution g of the Beltrami equation (1.1) with the given μ. By the Riemann theorem without loss of generality we may assume that $g(0)=0$ and $g(\mathbb{D})=\mathbb{D}$. Since f as well as g are locally quasiconformal in the punctured disk $\mathbb{D}\setminus\{0\}$, then by the uniqueness theorem for the quasiconformal mappings $f=h\circ g$ in $\mathbb{D}\setminus\{0\}$ where h is a conformal mapping in $\mathbb{D}\setminus\{0\}$. However, isolated singularities are removable for conformal mappings. Hence h can be extended by continuity to 0 and, consequently, f should be so. Thus, the obtained contradiction disproves the assumption (5.8).

5.16. Remark. Thus, Theorems 5.1 and 2.15 show that every of the conditions (2.17)–(2.22) in the existence theorems to the Beltrami equations (1.1) with the integral constraints (5.2) for non–decreasing convex functions Φ are not only sufficient but also necessary.

6 Historic comments and final remarks

To compare our results with erlier results of other authors we give a short survey.

The first investigation of the existence problem for degenerate Beltrami equations with integral constraints (4.18) as in Theorem 4.17 has been made by Pesin [Pe] who studied the special case where $\Phi(t) = e^{t^\alpha} - 1$ with $\alpha > 1$. Basically, Corollary 4.23 is due to Kruglikov [Kr]. David [Da] considered the existence problem with measure constraints

$$(6.1) \qquad |\{z \in D: \ K_\mu(z) \ > \ t\}| \ \leq \ \varphi(t) \qquad \forall\, t \in [1, \infty)$$

with special $\varphi(t)$ of the form $a \cdot e^{-bt}$ and Tukia [Tu] with the corresponding constraints in terms of the spherical area. Note that under the integral constraints (4.18) of the exponential type $\Phi(t) = \alpha e^{\beta t}$, $\alpha > 0$, the conditions of David and Tukia hold. Thus, the latter results sthrengthen the Pesin result.

By the well known John-Nirenberg lemma for the function of the class BMO (bounded mean oscillation) the David conditions are equivalent to the corresponding integral conditions of the exponential type, see e.g. [RSY$_1$]. More advanced results in terms of FMO (finite mean oscillation by Ignat'ev-Ryazanov) can be found in [RSY$_4$], [RSY$_6$] and [MRSY].

The next step has been made by Brakalova and Jenkins [BJ$_1$] who proved the existence of ACL homeomorphic solutions for the case of the integral constraints (4.2) as in Theorem 4.1 with $K_\mu(z)$ instead of $K^T_\mu(z, z_0)$ and with

$$(6.2) \qquad \Phi_{z_0}(t) \ \equiv \ \Phi(t) \ = \ \exp\left(\frac{\frac{t+1}{2}}{1 + \log\frac{t+1}{2}}\right) .$$

Note that, in the case [BJ$_1$], the condition (2.17) in Theorem 2.15, see also Corollary 4.12, can be easy verified by the calculations

$$(6.3) \qquad (\log \Phi(t))' \ = \ \frac{1}{2}\frac{\log\frac{t+1}{2}}{(1 + \log\frac{t+1}{2})^2} \ \sim \ \frac{1}{2}\frac{1}{\log t} \qquad \text{as } t \to \ \infty .$$

Moreover, it is easy to verify that $\Phi''(t) \geq 0$ for all $t \geq T$ under large enough $T \in (1, \infty)$ and, thus, Φ is convex on the segment $[T, \infty]$, see e.g. [Bou] and Remark 4.25.

Later on, Iwaniec and Martin have proved the existence of solutions in the Orlicz–Sobolev classes for the case where

$$(6.4) \qquad \Phi_{z_0}(t) \ \equiv \ \Phi(t) \ = \ \exp\left(\frac{pt}{1 + \log t}\right)$$

for some $p > 0$, see e.g. [IM$_1$]–[IM$_2$], for which

$$(6.5) \qquad (\log \Phi(t))' \ = \ \frac{p\log\ t}{(1 + \log t)^2} \ \sim \ \frac{p}{\log t} \qquad \text{as } t \to \ \infty ,$$

cf. Corollary 4.20. Note that in the both cases (6.2) and (6.4)

$$\Phi(t) \ \geq \ t^{\lambda} \qquad \forall\, t \ \geq \ t_{\lambda} \ \in \ [1, \infty) \ . \tag{6.6}$$

It is remarkable that in the case

$$\Phi_{z_0}(t) \ \equiv \ \Phi(t) \ = \ \exp \ pt \tag{6.7}$$

it was established uniqueness and factorization theorems for solutions of the Beltrami equations of the Stoilow type, see e.g. [AIM] and [Da].

Corollary 4.10 is due to Gutlyanskii, Martio, Sugawa and Vuorinen in [GMSV$_1$] and [GMSV$_2$] where they have established the existence of ACL homeomorphic solutions of (1.1) in $W^{1,s}_{loc}$, $s = 2p/(1+p)$, under $K_{\mu} \in L^p_{loc}$ with $p > 1$ for

$$\Phi_{z_0}(t) \ \equiv \ \Phi(t) := \ \exp H(t) \tag{6.8}$$

with $H(t)$ being a continuous non–decreasing function such that $\Phi(t)$ is convex and

$$\int\limits_1^{\infty} H(t)\, \frac{dt}{t^2} \ = \ \infty \ . \tag{6.9}$$

It was one of the most outstanding results in the field of criteria for the solvability of the degenerate Beltrami equations as it is clear from Theorem 5.1, see also Remark 5.16.

Subsequently, the fine theorems on the existence and uniqueness of solutions in the Orlich–Sobolev classes have been established under the condition (6.9) with the smooth H and the condition $tH'(t) \geq 5$, see Theorem 20.5.2 in the monograph [AIM], cf. Lemma 5.4, see also Remark 5.6 above. However, we have not found the work [GMSV$_2$] in the reference list of this monograph. The theorems on the existence and uniqueness of solutions in the class $W^{1,2}_{loc}$ have been established also before it under $K_{\mu}(z) \leq Q(z) \in W^{1,2}_{loc}$ in the work [MM].

Recently Brakalova and Jenkins have proved the existence of ACL homeomorphic solutions under (4.2), again with $K_{\mu}(z)$ instead of $K^T_{\mu}(z, z_0)$, and with

$$\Phi_{z_0}(t) \ \equiv \ \Phi(t) \ = \ h\left(\frac{t+1}{2}\right) \tag{6.10}$$

where they assumed that h is increasing and convex and $h(x) \geq C_{\lambda} x^{\lambda}$ for any $\lambda > 1$ with some $C_{\lambda} > 0$ and

$$\int\limits_1^{\infty} \frac{d\tau}{\tau h^{-1}(\tau)} \ = \ \infty \ , \tag{6.11}$$

see [BJ$_2$]. Note that the conditions $h(x) \geq C_{\lambda} x^{\lambda}$ for any $\lambda > 1$, in particular, under the above sub-exponential integral constraints, see (6.6), imply that K_{μ} is locally integrable with any degree $p \in [1, \infty)$, see (1.12).

Some of the given conditions are not necessary as it is clear from the results in Section 4 and from the following lemma and remarks.

6.12. Lemma. *There exist continuous increasing convex functions* $\Phi : [1,\infty) \to [1,\infty)$ *such that*

(6.13)
$$\int\limits_1^{\infty} \log \Phi(t)\ \frac{dt}{t^2}\ =\ \infty\ ,$$

(6.14)
$$\liminf_{t\to\infty}\ \frac{\log \Phi(t)}{\log t}\ =\ 1$$

and, moreover,

(6.15)
$$\Phi(t) \geq t \qquad \forall\ t \in [1,\infty)\ .$$

Note that the examples from the proof of Lemma 6.12 further can be extended to $[0,\infty]$ by $\Phi(t) = t$ for $t \in [0,1]$ with keeping all the given properties.

6.16. Remark. The condition (6.14) implies, in particular, that there exist no $\lambda > 1, C_\lambda > 0$ and $T_\lambda \in [1,\infty)$ such that

(6.17)
$$\Phi(t)\ \geq C_\lambda \cdot t^\lambda \qquad \forall\ t \geq T_\lambda\ .$$

Thus, in view of Lemma 6.12 and Theorem 4.17, no of the conditions (6.17) is necessary in the existence theorems for the Beltrami equations with the integral constraints of the type (4.18).

In addition, for the examples of Φ given in the proof of Lemma 6.12,

(6.18)
$$\limsup_{t\to\infty}\ \frac{\log\ \Phi(t)}{\log\ t}\ =\ \infty\ ,$$

cf. Proposition 6.26 further. Finally, all the conditions (2.17)–(2.22) from Theorem 2.15 hold simultaneously with (6.13) because the increasing convex function Φ is absolutely continuous.

Proof of Lemma 6.12. Further we use the known criterion which says that a function Φ is convex on an open interval I if and only if Φ is continuous and its derivative Φ' exists and is non-decreasing in I except a countable set of points in I, see e.g. Proposition 1.4.8 in [Bou].We construct Φ by induction sewing together pairs of functions of the two types $\varphi(t) = \alpha + \beta t$ and $\psi(t) = ae^{bt}$ with suitable positive parameters a, b and β and possibly negative α.

More precisely, set $\Phi(t) = \varphi_1(t)$ for $t \in [1,\gamma_1^*]$ and $\Phi(t) = \psi_1(t)$ for $t \in [\gamma_1^*,\gamma_1]$ where $\varphi_1(t) = t$, $\gamma_1^* = e$, $\psi_1(t) = e^{-(e-1)}e^t$, $\gamma_1 = e+1$. Let us assume that we already constructed $\Phi(t)$ on the segment $[1,\gamma_n]$ and hence that $\Phi(t) = a_n e^{b_n t}$ on the last subsegment $[\gamma_n^*,\gamma_n]$ of the segment $[\gamma_{n-1},\gamma_n]$. Then we set $\varphi_{n+1}(t) = \alpha_{n+1} + \beta_{n+1}t$ where the parameters α_{n+1} and β_{n+1} are found from the conditions $\varphi_{n+1}(\gamma_n) = \Phi(\gamma_n)$ and $\varphi'_{n+1}(\gamma_n) \geq \Phi'(\gamma_n - 0)$, i.e., $\alpha_{n+1} + \beta_{n+1}\gamma_n = a_n e^{b_n\gamma_n}$ and $\beta_{n+1} \geq a_n b_n e^{b_n\gamma_n}$. Let $\beta_{n+1} = a_n b_n e^{b_n\gamma_n}$, $\alpha_{n+1} = a_n e^{b_n\gamma_n}(1 - b_n\gamma_n)$ and choose a large enough $\gamma_{n+1}^* > \gamma_n$ from the condition

(6.19)
$$\log\left(\alpha_{n+1} + \beta_{n+1}\gamma_{n+1}^*\right)\ \leq\ \left(1 + \frac{1}{n}\right) log\ \gamma_{n+1}^*$$

and, finally, set $\Phi(t) \equiv \varphi_{n+1}(t)$ on $[\gamma_n, \gamma^*_{n+1}]$.

Next, we set $\psi_{n+1}(t) = a_{n+1}e^{b_{n+1}t}$ where parameters a_{n+1} and b_{n+1} are found from the conditions that $\psi_{n+1}(\gamma^*_{n+1}) = \varphi_{n+1}(\gamma^*_{n+1})$ and $\psi'_{n+1}(\gamma^*_{n+1}) \geq \varphi'_{n+1}(\gamma^*_{n+1})$, i.e.,

$$b_{n+1} = \frac{1}{\gamma^*_{n+1}} \log \frac{\alpha_{n+1} + \beta_{n+1}\gamma^*_{n+1}}{a_{n+1}} \tag{6.20}$$

and, taking into account (6.20),

$$b_{n+1} \geq \frac{\beta_{n+1}}{\alpha_{n+1} + \beta_{n+1}\gamma^*_{n+1}} . \tag{6.21}$$

Note that (6.21) holds if we take small enough $a_{n+1} > 0$ in (6.20). In addition, we may choose here $b_{n+1} > 1$.

Now, let us choose a large enough γ_{n+1} with $e^{-1}\gamma_{n+1} \geq \gamma^*_{n+1}$ from the condition that

$$\log \psi_{n+1}\left(e^{-1}\gamma_{n+1}\right) \geq e^{-1}\gamma_{n+1} , \tag{6.22}$$

i.e.,

$$\log a_{n+1} + b_{n+1}e^{-1}\gamma_{n+1} \geq e^{-1}\gamma_{n+1} . \tag{6.23}$$

Note that (6.23) holds for all large enough γ_{n+1} because $b_{n+1} > 1$ although $\log a_{n+1}$ can be negative.

Setting $\Phi(t) = \psi_{n+1}(t)$ on the segment $[\gamma^*_{n+1}, \gamma_{n+1}]$, we have that

$$\log \Phi(t) \geq t \qquad \forall\, t \in [e^{-1}\gamma_{n+1}, \gamma_{n+1}] \tag{6.24}$$

where the subsegment $[e^{-1}\gamma_{n+1}, \gamma_{n+1}] \subseteq [\gamma^*_{n+1}, \gamma_{n+1}]$ has the logarithmic length 1.

Thus, (6.15) holds because by the construction $\Phi(t)$ is absolutely continuous, $\Phi(1) = 1$ and $\Phi'(t) \geq 1$ for all $t \in [1, \infty)$; the equality (6.13) holds by (6.24); (6.14) by (6.15) and (6.19); (6.18) by (6.24).

6.25. Remark. Taking in the above construction in Lemma 6.12 $\beta_{n+1} = 1$ for all $n = 1, 2, \ldots$, $\alpha_{n+1} = b_n e^{b_n\gamma_n} - \gamma_n$ and arbitrary $\gamma^*_{n+1} > \gamma_{n+1}$ we obtain examples of absolutely continuous increasing functions Φ which are not convex but satisfy (6.13), as well as all the conditions (2.17)–(2.22) from Proposition 2.15, and (6.15).

The corresponding examples of non-decreasing functions Φ which are neither continuous, nor strictly monotone and nor convex in any neighborhood of ∞ but satisfy (6.13), as well as (2.17)–(2.22), and (6.15) are obtained in the above construction if we take $\beta_{n+1} = 0$ and $\alpha_{n+1} > \gamma_n$ such that $\alpha_{n+1} > \Phi(\gamma_n)$ and $\Phi(t) = \alpha_{n+1}$ for all $t \in (\gamma_n, \gamma^*_{n+1}]$, $\gamma^*_{n+1} = \alpha_{n+1}$.

6.26. Proposition. *Let $\Phi : [1, \infty) \to [1, \infty)$ be a locally integrable function such that*

$$\int\limits_1^{\infty} \log \Phi(t)\, \frac{dt}{t^2} = \infty . \tag{6.27}$$

Then

$$\limsup_{t\to\infty}\frac{\Phi(t)}{t^\lambda} = \infty \qquad \forall\ \lambda\in\mathbb{R}\ . \tag{6.28}$$

6.29. Remark. In particular, (6.28) itself implies the relation (6.18). Indeed, we have from (6.28) that there exists a monotone sequence $t_n\to\infty$ as $n\to\infty$ such that

$$\Phi(t_n)\ \geq\ t_n^n\,,\quad n=1,2,\ldots, \tag{6.30}$$

i.e.,

$$\frac{\log\Phi(t_n)}{\log t_n}\ \geq\ n,\ \ n=1,2,\ldots\ . \tag{6.31}$$

Proof of Proposition 6.26. It is sufficient to consider the case $\lambda>0$. Set $H(t)=\log\Phi(t)$, i.e., $\Phi(t)=e^{H(t)}$. Note that $e^x\geq x^n/n!$ for all $x\geq 0$ and $n=1,2\ldots$, because $e^x=\sum\limits_{n=0}^{\infty}x^n/n!$. Fix $\lambda>0$ and $n>\lambda$. Then $q:=\lambda/n$ belongs to $(0,1)$ and

$$\frac{H(t)}{t^q}\ \leq\ \left(\frac{\Phi(t)}{t^\lambda}\right)^{\frac{1}{n}}\cdot\sqrt[n]{n!}\ .$$

Let us assume that

$$C:=\ \limsup_{t\to\infty}\ \frac{\Phi(t)}{t^\lambda}\ <\ \infty\ . \tag{6.32}$$

Then

$$\int\limits_{\triangle}^{\infty}H(t)\ \frac{dt}{t^2}\ <\ 2\sqrt[n]{Cn!}\int\limits_{\triangle}^{\infty}\frac{dt}{t^{2-q}}\ =\ -\ \frac{2}{1-q}\ \frac{\sqrt[n]{Cn!}}{t^{1-q}}\ \Big|_{\triangle}^{\infty}=$$

$$=\ \frac{2}{1-q}\ \frac{\sqrt[n]{Cn!}}{\triangle^{1-q}}\ <\ \infty$$

for large enough $\triangle>1>0$. The latter contradicts (6.27). Hence the assumption (6.32) was not true and, thus, (6.28) holds for all $\lambda\in\mathbb{R}$.

6.33. Remark. Lemma 6.12 shows that, generally speaking, lim sup in (6.28) cannot be replaced by lim for an arbitrary $\lambda>1$ under the condition (6.27) even if Φ is continuous, increasing and convex.

Applications of strong ring solutions to the theory of boundary problems for the Beltrami equations will be published elsewhere, see e.g. [Dy], cf. [RS] and [Lo].

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